THE BOCHNER MEASURE AND HOLOMORPHIC EXTENSION OF ELEMENTARY SPHERICAL FUNCTIONS

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Introduction.

Let G be a noncompact simple linear Lie group, and let K be a maximal compact subgroup. Let $\widehat{G}$ denote the set of equivalence classes of irreducible unitary representations of G and let $\widehat{G}_s$ denote those containing a trivial K-type. We recall several results from the work of Harish-Chandra. Any $(\pi, \mathcal{H}_\pi) \in \widehat{G}_s$ contains the trivial K-type with multiplicity one; let $v_0 \in \mathcal{H}_\pi$ be one of norm one. Denote by φ_π the matrix coefficient

$$\varphi_\pi(g) = \langle \pi(g) v_0, v_0 \rangle.$$

Then φ_π is in $C^\omega(G//K), \varphi_\pi(e) = 1,$ and φ_π is a positive definite function on G. One can give an explicit realization of such $(\pi, \mathcal{H}_\pi)$ by using induced representations, and in this way enlarge the parameter domain for these functions. So let $P = MAN$ be a minimal parabolic subgroup with $M \subset K$. Set $\pi_{\sigma,\lambda} = \mathrm{Ind}_P^G(\sigma \otimes \chi_\lambda \otimes \mathbf{1})$ acting on $\mathcal{H}_{\sigma,\lambda}$ where $(\sigma, W_\sigma) \in \widehat{M}$, and $\chi_\lambda \in \mathrm{Hom}(A, \mathbb{C}^x) \cong \mathfrak{a}_\mathbb{C}^*$. For $\sigma = \mathbf{1}, \pi_\lambda := \pi_{\mathbf{1},\lambda}$ contains the trivial K type with multiplicity one. While, for $\lambda \in \mathfrak{a}_\mathbb{C}^*$, π_λ need not in general be irreducible nor unitarizable, we do have $\widehat{G}_s \hookrightarrow \{(\pi_\lambda, \mathcal{H}_\lambda) | \lambda \in \mathfrak{a}_\mathbb{C}^*\}$ and, after incorporating the intertwining operator into the inner product, this gives unitary realizations. The explicit form of the inner product on $\mathcal{H}_\lambda$ gives

$$\varphi_\lambda(g) := \varphi_{\pi_\lambda}(g) = \int_K a(gk)^{\lambda - \rho} \, dk$$

which is clearly entire on $\mathfrak{a}_\mathbb{C}^*$. If we set

$$\varphi(x : \lambda) := \varphi_\lambda(x), x \in G, \lambda \in \mathfrak{a}_\mathbb{C}^*$$

then φ is real analytic in "x"and entire in "λ".

In [KS-I] we constructed a universal domain in the complexification, $G_\mathbb{C}$, of G and we showed that the matrix coefficients obtained from K-finite vectors of any irreducible Banach representation of G have holomorphic extension to this domain. When one of the vectors is the trivial K-type this domain naturally maps to one in $G_\mathbb{C}/K_\mathbb{C}$. In [KÓS] we showed that this domain, denoted Ξ, is a maximal G-invariant domain with this property, although not a maximal $K_\mathbb{C}$-invariant one. Consequently $\varphi(\cdot : \lambda)$ has holomorphic extension to Ξ. This domain is determined mainly by imposing two constraints, namely the holomorphic continuation of the A component from the Iwasawa decomposition and the construction of non-unitary principal series in conjunction with the subrepresentation theorem. In this note we shall give an alternative justification of the holomorphic extension in the specific setting of spherical unitary representations and elementary spherical functions. The novelty of this analysis includes the definition of a *spectral* version of the Abel transform $f \to F_f$, and an integral formula which we make explicit for split rank one groups and from which we derive several consequences.

The starting point is the comment above that φ_π is a positive definite function. The characterization of continuous positive definite functions on $\mathbb{R}$ by means of their Fourier

transforms and more importantly their relevance to such areas as spectral theory and probability was discovered by Bochner. While there have been extensions of the characterization theorem to locally compact abelian groups and to some extent non-abelian groups, they have not played a central role in the harmonic analysis on these groups. Yet a central theme in the development of harmonic analysis on semisimple Lie groups has been to examine the objects of harmonic analysis–matrix coefficients and characters–on certain abelian (or almost abelian) subgroups. It seems natural then, following Bochner's lead, to try to characterize the abelian Fourier transforms of these objects and hopefully to find some uses of them in harmonic analysis. This paper is the result of our first look at this problem.

The setting which seemed most amenable to this investigation is the non-unitary principal series of representations induced from a minimal parabolic subgroup MAN. Here the abelian group A enters in three well-known ways: part of the starting data is a non-unitary character of A; the representation space can be realized on functions on a subgroup V on which A acts in a reasonable way; the growth properties of K-finite matrix coefficients are determined by their restrictions to A.

In order to apply abelian harmonic analysis on A in the most direct way to matrix coefficients of principal series representations, we shall use a minor modification of the "non-compact picture" of non-unitary principal series which is built from an A-invariant measure on V. The measure is constructed in §3 and the realization in §4. The Fourier transform, i.e., the Bochner measure, of matrix coefficients (not necessarily K-finite) of unitary principal series follows easily from this realization and is included in §4. In §5 we consider the $\mathbb{R}$-rank 1 case and show that the Bochner measure can be computed for non-unitary principal series, K-finite coefficients provided $|\mathrm{Re}\ \lambda| < |\rho|$. For the zonal spherical functions, in §6 we compute the Bochner measure explicitly for split rank one groups, obtaining an integral representation which seems new even from a classical special function point of view. In §7 we give four applications of the Bochner measure. In one we derive the Harish-Chandra expansion directly from the integral formula without any differential equations. Another is to define a *spectral* F_f with properties in the spectral parameter analogous to those of F_f in the group parameter.

§1. Laplace transforms.

In classical abelian harmonic analysis there is a well known relationship, due to Paley and Wiener, between the holomorphic extension of functions to tubular neighborhoods and the exponential decay of their Fourier-Laplace transform. This will be examined in the context of spherical functions. We shall identify spherical functions as Laplace transforms on $\widehat{A} \cong i\mathfrak{a}^* \cong \mathfrak{a}^*$. For this purpose we briefly review some standard facts on Laplace transforms.

Let V be a finite dimensional real vector space and V^* its dual. We denote by $V_{\mathbb{C}} = V + iV$ the complexification of V. Let μ be a positive Borel measure on V^* and set

$$D_\mu = \{x \in V\colon \int_{V^*} e^{-\alpha(x)}\, d\mu(\alpha) < \infty\}.$$

Shortly we shall assume that μ is *admissible*, i.e., $D_\mu \neq \emptyset$ (thus, in particular, that μ is a Radon measure). The *Laplace transform of* μ is then defined by

$$\mathcal{L}\mu\colon D_\mu \to \mathbb{R}_0^+, \quad x \mapsto \int_{V^*} e^{-\alpha(x)}\, d\mu(\alpha).$$

Note that $\mathcal{L}\mu$ automatically extends to a holomorphic function on the tube domain $T_{D_\mu} = iV + \operatorname{int} D_\mu \subseteq V_{\mathbb{C}}$. We denote the extension also by $\mathcal{L}\mu$. We shall need the following results on Laplace transforms.

Proposition 1.1. *Let μ be an admissible positive Borel measure on the dual V^* of a finite dimensional real vector space V. Then*

(i) *the domain of definition D_μ of the Laplace transform $\mathcal{L}\mu$ is a convex subset of V;*

(ii) *suppose that μ is finite and that there is an open connected set $\Omega \subseteq V$ with $0 \in \Omega$ such that $\mathcal{L}\mu$ has an extension to a holomorphic function on the tube domain $T_\Omega = iV + \Omega$, then we have $\Omega \subseteq D_\mu$, i.e., on T_Ω the extension is represented by*

$$T_\Omega \to \mathbb{C}, \quad z \mapsto \int_{V^*} e^{-\alpha(z)}\, d\mu(\alpha).$$

Proof. (i) [Ne, Prop. V.4.3].
(ii) [Ri, p. 311]. ■

Let $A = \mathbb{R}^n$ be a simply connected abelian real Lie group. We identify the unitary dual $\widehat{A}$ of A with $(\mathbb{R}^n)^*$ by means of the isomorphism,

$$(\mathbb{R}^n)^* \to \widehat{A}, \quad \alpha \mapsto (x \mapsto e^{i\alpha(x)}).$$

Let $\mathcal{B}(\widehat{A})$ denote the Borel σ-algebra on $\widehat{A}$ and for $(\pi, \mathcal{H})$ a unitary representation of A let $P(\mathcal{H})$ be the set of selfadjoint projections on $\mathcal{H}$. Then there exists a spectral measure

$$E\colon\ \mathcal{B}(\widehat{A}) \to P(\mathcal{H})$$

such that

$$(\forall x \in A)\ \ \pi(x) = \int_{\widehat{A}} e^{i\alpha(x)}\ dE(\alpha).$$

We are now going to apply the above considerations to spherical functions i.e. as matrix coefficients of the K-fixed vector of spherical principal series representations. Recall from [KS-I] the definition of the set $\Omega \subset \mathfrak{a}$ (or see below) and from Prop. 4.1 therein that φ_λ holomorphically extends to $Gexp(i\Omega)K_{\mathbb{C}}$, and from Theorem 4.2 to $K_{\mathbb{C}}exp(i2\Omega)K_{\mathbb{C}}$.

Lemma 1.2. *Let $\lambda \in i\mathfrak{a}^*$ and φ_λ be the associated spherical function on G/K. Then there exists a Radon probability measure μ on $\mathfrak{a}^*$ such that*

$$(\forall a \in exp\, i2\Omega) \qquad \varphi_\lambda(a^{-1}) = \int_{\mathfrak{a}^*} e^{i\alpha(\log a)}\ d\mu(\alpha).$$

Proof. We consider the unitary representation $(\pi, \mathcal{H}) = (\pi_\lambda \mid_A, \mathcal{H}_\lambda)$ of A. The probability measure μ in question is then given by $\mu = E_{v_0,v_0}$ with $v_0 = \mathbf{1}_K$. In particular, we get that

$$(\forall a \in A) \qquad \varphi_\lambda(a^{-1}) = \int_{\mathfrak{a}^*} e^{i\alpha(\log a)}\ d\mu(\alpha).$$

Now the assertion of the Theorem follows from [KS-I Theorem 4.2] and Proposition 1.1(ii). ∎

Remark 1.3. For $\lambda \in i\mathfrak{a}^*$, as mentioned before, the spherical function φ_λ is a continuous positive definite function with $\varphi_\lambda(e) = 1$. In particular on the abelian group $A \cong \mathfrak{a}$, $\varphi_\lambda|A$ has these properties. Hence the existence of a probability measure μ on $\mathfrak{a}^*$ such that

$$\varphi_\lambda(a^{-1}) = \int_{\mathfrak{a}^*} e^{i\alpha(\log a)}\ d\mu(\alpha)$$

holds for all $a \in A$ is also a consequence of Bochner's Theorem. (In fact, Bochner used his result to give a proof of the spectral theorem.) However, the validity of the extension to $exp\, 2i\Omega$ of the Bochner integral representation is far from obvious. For example, the geometry of the tube reflects growth estimates of the Bochner measure and these are usually difficult to compute. All these remarks are illustrated in considerable detail in the case of rank 1 groups in subsequent sections.

In view of Lemma 1.2 and Proposition 1.1(ii) every restricted spherical function $\varphi_\lambda \circ \exp_{\mathrm{A}}$, has a holomorphic extension to a tube domain

$$T_{\lambda,\max} = \mathfrak{a} + i\omega_\lambda$$

which is maximal with respect to $\omega_\lambda \subseteq \mathfrak{a}$ open, convex, $0 \in \omega_\lambda$ and ω_λ is $\mathcal{W}_{\mathfrak{a}}$-invariant. In [K-S I Th. 4.2] we show that $2\Omega \subset \omega_\lambda$ and in §5 for all real rank one groups $2\Omega = \omega_\lambda$ where $2\Omega = \{X \in \mathfrak{a}\colon (\forall \alpha \in \Sigma)\ |\alpha(X)| < \pi\}$.

§2. The Technique.

Let X be a locally compact space and let A be a locally compact group with a continuous proper action $A \times X \to X$, denoted $(a, x) \to a \cdot x$. Let dx be a measure on X and da a left Haar measure on A. We shall make some assumptions to specialize the presentation for the simple applications we have in mind.

First concerning the action, we shall assume that A acts without fixed points so that each orbit is homeomorphic to A; we shall require that the orbit space $A\backslash X$ is paracompact; and we want the measure dx to be A invariant. We will be concerned only with the case in which A is abelian and isomorphic to a finite dimensional real vector space. Under such conditions, there is a unique measure $d\omega$ on $A\backslash X$ and the usual "double integration" formula for reasonable f ([B] p. 44).

$$\int_X f(x)dx = \int_{A\backslash X} \int_A f(a \cdot x)da\ d\omega.$$

Moreover, if we make $A \times A\backslash X$ an A-space by translation in the A coordinate, then X and $A \times A\backslash X$ are homeomorphic A-spaces ([B] p. 73). If f is a function on X, by abuse of notation, we will let f also denote the corresponding function on $A \times A\backslash X$. The integration formula then becomes

$$\int_X f(x)\ dx = \int_{A\times A\backslash X} f(a, \omega)da \times d\omega$$

Since each orbit type is basically A we can transfer A harmonic analysis to each orbit. Thus, if $\widehat{A}$ denotes the unitary character group of A with dual measure, for reasonable functions f and g on X we have the familiar abelian results:

$$\text{if } \chi_\lambda \in \widehat{A},\ \widehat{f}(\lambda, \omega) = \int_A f(t, \omega)\chi_\lambda(t)^{-1}da; \tag{1'}$$

for $a \in A$ with the usual transport of structure set $L_a^* f(t, \omega) = f(a^{-1}t, \omega)$, then

$$\widehat{L_a^* f}(\lambda, \omega) = \chi_\lambda(a^{-1})\widehat{f}(\lambda, \omega); \tag{2'}$$

and the Parseval formula

$$\int_A f(t, \omega)\overline{g(t, \omega)}da = \int_{\widehat{A}} \widehat{f}(\lambda, \omega)\overline{\widehat{g}(\lambda, \omega)}d\lambda. \tag{3'}$$

For our purposes, we shall need to consider the usual action L_a^* on functions twisted by a character. This causes minor changes in $(1)', (2)'$ and $(3)'$. So suppose χ_v is a continuous character of A (not necessarily unitary) and define

$$a \cdot f(t,\omega) = \chi_v(a) L_a^* f(t,\omega).$$

For the function f_v defined by $f_v(t,\omega) = \chi_v(t^{-1}) f(t,\omega)$ we have

$$\begin{aligned}(a \cdot f)_v(t,\omega) &= \chi_v(t^{-1})\chi_v(a) L_a^* f(t,\omega)\\ &= \chi_v(a^{-1}t)^{-1} L_a^* f(t,\omega)\\ &= L_a^* f_v(t,\omega).\end{aligned}$$

We shall call the contragredient to χ_v the character $\chi_{v'}$ defined by $\chi_v \overline{\chi_{v'}} = 1$. For this A-action the modified Fourier transform defined by

$$\widetilde{f}(\lambda,\omega) := \widehat{f}_v(\lambda,\omega) \tag{2.1}$$

has the property

$$\widetilde{a \cdot f}(\lambda,\omega) = \chi_\lambda(a^{-1}) \widetilde{f}(\lambda,\omega), \tag{2.2}$$

and the Parseval formula

$$\begin{aligned}\int_A f(a,\omega)\overline{g(a,\omega)}\,da &= \int_{\widehat{A}} \widehat{f}_v(\lambda,\omega)\overline{\widehat{g}_{v'}(\lambda,\omega)}\,d\lambda \\ &= \int_{\widehat{A}} \widetilde{f}(\lambda,\omega)\overline{\widetilde{g}(\lambda,\omega)}\,d\lambda\end{aligned} \tag{2.3}$$

provided f_v and $g_{v'}$ satisfy reasonable growth conditions.

§3. The invariant measure on V.

An interesting example of such an A-space (in fact our motivation) arises in semisimple Lie groups. We shall briefly recall some basic facts about these groups. (See [H1]).

Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$. Fix a Cartan involution and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding decomposition. Choose a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. The adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ is diagonalizable; let Σ denote the set of non-zero real eigenvalues and $\mathfrak{g}_\alpha, \alpha \in \Sigma$, the corresponding eigenvectors. Fix a lexicographic order on Σ and let Σ^+ be the positive elements. If subalgebras of $\mathfrak{g}$ are defined by $\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha, \mathfrak{v} = \sum_{-\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ and $\mathfrak{m}_1 =$ zero eigenspace, then $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m}_1 \oplus \mathfrak{v}$. Let V, N, M_1, A denote the corresponding subgroups. Then $M_1 = MA$ where $M \subset K$ and centralizes A. Moreover, V, N and A are homeomorphic to their respective Lie algebras.

Since $\mathfrak{v}$ consists of $\mathfrak{a}$-eigenspaces, the group A acts on it and hence on V. Denote this action on V (conjugation) by $(a, v) \to a \cdot v$ or $\delta_a(v)$, and on functions by $\delta_a^* f(v) = f(\delta_{a^{-1}}(v))$. There are two aspects of this action which we must examine before using §2: (1) the obvious measure on V, Haar measure dv, is not invariant but only relatively invariant; (2) the action of A on V is not free. We shall remedy the first and define away the second.

The multiplier for the A action on dv is given by the character $\chi_{-2\rho}$, where $2\rho = \sum_{\alpha\in\Sigma^+} \alpha$, i.e. for $f \in C_c(V)$

$$\delta_{a*} dv[f] := dv[\delta_a^* f] = \chi_{-2\rho}(a) dv[f] = \chi_{2\rho}(a^{-1}) dv[f].$$

We will use $\chi_{-\rho}$ to construct an invariant measure on V.

The eigenspace decomposition of $\mathfrak{g}$ has an analog for G, namely the Bruhat big cell, such that except for a closed set of lower dimension $G = VMAN$ with the unique A-component denoted $a(g)$. Let $w \in G$ represent the Weyl group element that sends Σ^+ to $-\Sigma^+$. Then w^2 represents the identity, $wNw^{-1} = V$ and for $v \neq e, w^{-1}v \in VMAN$ ([Kn-S]). We extend $\chi_{-\rho}$ to a function $\xi_{-\rho}$ defined a.e. on G by

$$\xi_{-\rho}(x) := \chi_{-\rho}(a(x)).$$

Lemma 3.1. *If x is in $VMAN$ and a is in A then ax and xa are in $VMAN$. Also, $\xi_{-\rho}(ax) = \chi_{-\rho}(a)\xi_{-\rho}(x)$ and $\xi_{-\rho}(xa) = \xi_{-\rho}(x)\chi_{-\rho}(a)$.*

Proof. Obvious. ■

Define $\chi_{w\rho}(a) = \chi_\rho(w^{-1}aw)$. Then $\chi_{w\rho}$ is a character and

Lemma 3.2. *$\chi_{w\rho}$ is contragredient to χ_ρ.*

Proof. Since χ_ρ is $\mathbb{R}$-valued, this follows from $w\Sigma^+ = \Sigma^-$. ■

Proposition 3.3. *The measure $\xi_{-\rho}(w^{-1}v)dv$ on V is A-invariant.*

Proof. Let $f \in C_c(V)$. First, notice that

$$\begin{aligned}\delta_a^*(\xi_{-\rho}(w^{-1}\cdot)f)(v) &= \xi_{-\rho}(w^{-1}a^{-1}va)f(a^{-1}\cdot v)\\ &= \chi_{-\rho}(w^{-1}a^{-1}w)\xi_{-\rho}(w^{-1}v)\chi_{-\rho}(a)\delta_a^*f(v)\\ &= \chi_{-w\rho}(a^{-1})\chi_{-\rho}(a)\xi_{-\rho}(w^{-1}v)\delta_a^*f(v)\\ &= \chi_{-2\rho}(a)\xi_{-\rho}(w^{-1}v)\delta_a^*f(v).\end{aligned}$$

Here we have used Lemmas 3.1 and 3.2. Now

$$\begin{aligned}\delta_{a^*}(\xi_{-\rho}(w^{-1}\cdot)dv)[f] &= \xi_{-\rho}(w^{-1}\cdot)dv[\delta_a^*f]\\ &= dv[\xi_{-\rho}(w^{-1}\cdot)\delta_a^*f]\\ &= dv[\chi_{2\rho}(a)\delta_a^*(\xi_{-\rho}(w^{-1}\cdot)f)]\\ &= \chi_{2\rho}(a)\delta_{a^*}dv[\xi_{-\rho}(w^{-1}\cdot)f]\\ &= \chi_{2\rho}(a)\chi_{-2\rho}(a)dv[\xi_{-\rho}(w^{-1}\cdot)f]\\ &= \xi_{-\rho}(w^{-1})dv[f].\end{aligned}$$

■

To obtain a space on which A acts freely we shall discard a finite number of lower dimensional subspaces of $\mathfrak{v}$ leaving an open dense subset $\mathfrak{v}'$ which can be identified with a similar set $V' \subseteq V$. For the order Σ^+ let $\alpha_1, \cdots \alpha_\ell$ be a basis of simple restricted roots. For each i let $W_i = \sum_{t\alpha_i \in \Sigma^+} \oplus \mathfrak{g}_{t\alpha_i}$. Each W_i has a unique complementary subspace which is a sum of eigenspaces, say W_i^c. Let $W^c = \bigcup_{i=1}^{\ell} W_i^c$ and set $\mathfrak{v}' = \mathfrak{v} \backslash W^c$.

Lemma 3.4. *A acts freely on $\mathfrak{v}'$.*

Proof. If for each i, P_i denotes projection along W_i^c of $\mathfrak{v} \to W_i$, then $\mathfrak{v}'$ can be characterized as $\bigcap_{i=1}^{\ell} \{x \in \mathfrak{v} | P_i x \neq 0\}$. Now each $a \in A$ has all non-zero eigenvalues and commutes with each P_i, therefore $\mathfrak{v}'$ is A-invariant. Also, $\alpha_1, \cdots, \alpha_\ell$ form a basis for the dual space $\mathfrak{a}'$, so for each $a \in A, a \neq 1$, there is an i with $\chi_{\alpha_i}(a) \neq 1$. If $x \in \mathfrak{v}'$ were fixed by a then $P_i x = P_i a \cdot x = a \cdot P_i x$. But $P_i x \neq 0$ and on W_i the element a acts by $\chi_{\alpha_i}(a)$ or $\chi_{\alpha_i}(a)^2$ both of which are not 1. Thus the action is free.

We let V' be the corresponding dense open set in V and restrict $\xi_{-\rho}(w^{-1}v)dv$ to V'. This serves as the (X, dx) of §2.

Remarks. (1) If dim $\mathfrak{a} = 1$ then $\mathfrak{v}' = \mathfrak{v} \backslash \{0\}$ and so is maximal. If dim $\mathfrak{a} > 1$ the maximal subset on which A acts freely is not necessarily $\mathfrak{v}'$ but can be constructed. As we have no use for it here and the construction is standard, we omit it.

(2) M also acts on V by $(m,x) \to m \cdot x = mxm^{-1}$. One can show that M acts on V' and the measure $\xi_{-\rho}(w^{-1}v)dv$ is M-invariant. However, for the elementary spherical functions we shall not need this.

(3) It is worth noting that all that is done here can also be done for degenerate principal series induced off a maximal parabolic with $\mathfrak{n}$ a prehomogeneous vector space of parabolic type.

§4. The Bochner measure - unitary induction.

We recall briefly the construction of the principal series representations of G induced from the minimal parabolic subgroup MAN. We shall use as Hilbert space for these representations, the "noncompact picture" consisting of vector valued square integrable functions on V. Since the A-invariant measure is absolutely continuous with respect to Haar measure on V', a standard procedure provides an alternate realization of the principal series.

Let $(\sigma, W_\sigma) \in \widehat{M}$, and let χ_λ be a character of A, not necessarily unitary. In the "induced picture" one uses

$$C^\infty(G; \sigma \otimes \chi_\lambda) = \{f : G \to W_\sigma \mid f \text{ is } C^\infty \text{ and } f(gman) = \sigma(m)^{-1}\chi_{\lambda+\rho}(a)^{-1}f(g)\}$$

and has G act by left translation. From the Iwasawa decomposition, f is determined by its values on K and one takes as norm the L^2 norm of f restricted to K. From the Bruhat big cell, f is also determined by its values on V. In this "non-compact picture" the representation space $\mathcal{H}(\sigma, \chi)$ consists of the closure of $C_c^\infty(V, W_\sigma)$ in the norm

$$\int_V \|f(v)\|_\sigma^2 e^{2\mathrm{Re}\lambda(H(v)} dv$$

where $v = \kappa(v) \exp H(v)n(v)$ from the Iwasawa decomposition and $e^{2\mathrm{Re}\lambda H(v)} = |\chi_\lambda(\exp H(v))|^2$. The G action in this realization is given by

$$U(\sigma, \lambda)(g)f(x) = \sigma(m(g^{-1}x))^{-1}\chi_{\lambda+\rho}(a(g^{-1}x))^{-1}f(v(g^{-1}x))$$

where $g^{-1}x$ has been expressed in $VMAN$ coordinates.

Let χ'_λ be the contragredient character to χ_λ and form $\mathcal{H}(\sigma, \lambda')$. The mapping $f \to \chi_\lambda^2(\exp H(x))f(x)$ is an isometry of $\mathcal{H}(\sigma, \lambda)$ onto $\mathcal{H}(\sigma, \lambda')$. Moreover $\mathcal{H}(\sigma, \lambda)$ and $\mathcal{H}(\sigma, \lambda')$ are naturally dual via the pairing

$$(f, g) = \int_V < f(v), g(v) >_\sigma dv.$$

In addition, for f in $\mathcal{H}(\sigma, \lambda)$, g in $\mathcal{H}(\sigma, \lambda')$ and y in G, the matrix coefficient $c_{f,g}(y)$ is defined by

$$c_{f,g}(y) = (U(\sigma, \lambda)(y)f, g)$$

and one has the identity

$$(U(\sigma, \lambda)(y)f, g) = (f, U(\sigma, \lambda')(y^{-1})g).$$

The A-invariant measure on V' is $\xi_{-\rho}(w^{-1}v)dv$. Define a Hilbert space $\widetilde{\mathcal{H}}(\sigma, \lambda)$ to consist of the closure of $C_c^\infty(V, W_\sigma)$ in the norm

$$\int_V \|f(v)\|_\sigma^2 e^{2\mathrm{Re}\lambda H(v)} \xi_{-\rho}(w^{-1}v)dv.$$

The map $T : \mathcal{H}(\sigma,\lambda) \to \widetilde{\mathcal{H}}(\sigma,\lambda)$ given by

$$Tf(x) = \xi_{-\rho}(w^{-1}x)^{-1/2} f(x), x \neq e$$

has inverse

$$T^{-1}f(x) = \xi_{-\rho}(w^{-1}x)^{1/2} f(x)$$

and establishes an isomorphism between these spaces. There is then an equivalent realization of the principal series $\widetilde{U}(\sigma,\lambda)(\cdot)$ on $\widetilde{\mathcal{H}}(\sigma,\lambda)$ which we refer to as the "A-adapted" realization and is given by

$$\widetilde{U}(\sigma,\lambda)(\cdot) = T \circ U(\sigma,\lambda)(\cdot) \circ T^{-1}.$$

A straightforward computation using Lemma 3.1 and 3.2 gives

$$\widetilde{U}(\sigma,\lambda)(a)f(x) = \chi_\lambda(a)\delta_a^* f(x), a \in A, \tag{4.1}$$

$$\widetilde{U}(\sigma,\lambda)(m)f(x) = \sigma(m)\delta_m^* f(x), m \in M. \tag{4.2}$$

Furthermore, $\widetilde{\mathcal{H}}(\sigma,\lambda)$ and $\widetilde{\mathcal{H}}(\sigma,\lambda')$ are in natural duality via

$$(f,g) = \int_{V'} < f(x), g(x) >_\sigma \xi_{-\rho}(w^{-1}x)dx, \tag{4.3}$$

and one has

$$(\widetilde{U}(\sigma,\lambda)(y)f,g) = (f, \widetilde{U}(\sigma,\lambda')(y^{-1})g). \tag{4.4}$$

The A-adapted realization gives a new integral representation for matrix coefficients with close connections with abelian harmonic analysis. Using the construction from §2 we take $f_\lambda(t,\omega) = \chi_\lambda(t^{-1})f(t,\omega)$ with modified Fourier transform $\widetilde{f}(\upsilon,\omega) = \widehat{f_\lambda}(\upsilon,\omega)$, and similarly with the contragredient χ'_λ and g we obtain $\widetilde{g}$.

Theorem 4.1. *Let χ_λ be a unitary character of A and let $\dim \mathfrak{a} = \ell$. Let σ be a finite dimensional unitary representation of M. Let f be in $\widetilde{\mathcal{H}}(\sigma,\lambda)$ and g be in $\widetilde{\mathcal{H}}(\sigma,\lambda')$. Then, for a in A*

$$c_{f,g}(a) = \int_{\mathbb{R}^\ell} e^{-i\upsilon \log a} m(\lambda, \upsilon) d\upsilon. \tag{4.5}$$

Here $m(\lambda, \upsilon)$ is a function depending on f and g and given by

$$m(\lambda, \upsilon) = \int_{A\backslash V'} < \widetilde{f}(\upsilon, \omega), \widetilde{g}(\upsilon, \omega) >_\sigma d\omega. \tag{4.6}$$

Remarks. We shall refer to the measure $m(\lambda, \upsilon)d\upsilon$ as the Bochner measure for $c_{f,g}$. Since $\widetilde{U}$ is unitary the matrix coefficients $c_{f,f}$ are positive definite functions on G, a fortiori on A. Hence Bochner's theorem assures the existence of a non-negative measure with this property while (4.5) shows it is absolutely continuous with respect to Haar measure and (4.6) gives a formula for it.

Proof. We begin with (4.3)

$$\begin{aligned}
c_{f,g}(a) &= (\widetilde{U}(\sigma, \lambda)(a) f, g) \\
&= \int_{V'} < \widetilde{U}(\sigma, \lambda)(a) f(x), g(x) >_\sigma \xi_{-\rho}(w^{-1}x) dx \\
&= \int_{A \times A\backslash V'} < \widetilde{U}(\sigma, \lambda)(a) f(h, \omega), g(h, \omega) >_\sigma dh \times dw \\
&= \int_{A \times A\backslash V'} < \chi_\lambda(a) \delta_a^* f(h, \omega), g(h, \omega) >_\sigma dh \times d\omega
\end{aligned}$$

We continue with a formal application of Parseval's formula (2.3) and (2.2) to get

$$\begin{aligned}
c_{f,g}(a) &= \int_{\widehat{A} \times A\backslash V'} \chi_\upsilon(a)^{-1} < \widetilde{f}(\upsilon, \omega), \widetilde{g}(\upsilon, \omega) >_\sigma d\upsilon \times d\omega \\
&= \int_{\mathbb{R}^\ell} e^{-i\upsilon \log a} m(\lambda, \upsilon) d\upsilon.
\end{aligned}$$

To justify the use of (2.3), we observe that f, $\widetilde{U}f$ and g have square integrable σ-norms with respect to $\xi_{-\rho}(w^{-1}x)dx$. Hence, a.e. $d\omega$, $\widetilde{U}f(\cdot, \omega)$ and $g(\cdot, \omega)$ have square integrable σ-norms relative to dh. Since $a \in A$ acts by $\chi_\lambda(a^{-1})$ and (2.3) is valid for each component of the vector the result follows. ■

Remarks. The unitarity of χ_λ was used only to justify (2.3). For non-unitary χ_λ, whenever (2.3) can be used there is such an integral formula. Also, notice that f and g need not be K-finite.

§5. Non-unitary principal series.

We assume in this section that dim $\mathfrak{a} = 1$. Then $\mathfrak{a} = \mathrm{cl}(\mathfrak{a}^+ \cup w\mathfrak{a}^+ w^{-1})$. If λ and υ are $\mathbb{C}$-valued linear forms on $\mathfrak{a}$ we say $\lambda > \upsilon$ if $\mathrm{Re}(\lambda - \upsilon)(\mathfrak{a}^+) > 0$.

Theorem 5.1. *Let χ_λ be a character of A with $Re\lambda \geq 0$ and both $\rho > \lambda$ and $\rho > -\lambda'$. Let σ be a finite dimensional unitary representation of M. Let f in $\widetilde{\mathcal{H}}(\sigma, \lambda)$ and g in $\widetilde{\mathcal{H}}(\sigma, \lambda')$ be smooth vectors. Then*

$$c_{f,g}(a) = \int\limits_{\mathbb{R}} e^{-i\upsilon \log a} m(\lambda, \upsilon) d\upsilon,\ a \in A \tag{5.1}$$

where

$$m(\lambda, \upsilon) = \int\limits_{A\backslash V'} < \widetilde{f}(\upsilon, \omega), \widetilde{g}(\upsilon, \omega) >_\sigma d\omega \tag{5.2}$$

Proof. As remarked in §4 it is enough to justify the use of (2.3). Now it is well known that the intertwining operator between the compact and non-compact realization $\mathcal{H}(\sigma, \lambda)$ is given by

$$\mathrm{Ih}(x) = e^{-(\lambda+\rho)H(x)} h(\kappa(x)),\ x \in V.$$

Following I with T we may suppose f (and similarly g) is of the form

$$\xi_{-\rho}(w^{-1}x)^{-1/2} e^{-(\lambda+\rho)H(x)} h(\kappa(x))$$

with h a K-finite W_σ-valued function on K. In particular, each component of h is bounded. Now, $\widetilde{U}(\sigma, \lambda)(a) f(x) = \chi_\lambda(a) \xi_{-\rho}(w^{-1}a^{-1}xa)^{-1/2} e^{-(\lambda+\rho)H(a^{-1}xa)} h(\kappa(a^{-1}xa))$. Since h has bounded components, for the purpose of estimates it may be ignored. Choose a cross-section $A\backslash V' \to V'$ and denote a fixed element in the range by ω. Recall the definition from §2 of f_λ and $g_{\lambda'}$, i.e., $f_\lambda(t\omega t^{-1}) = \chi_\lambda(t)^{-1} f(t, \omega)$ and $g_{\lambda'}(t\omega t^{-1}) = \chi_{\lambda'}(t)^{-1} g(t, \omega)$· We shall show that the Parseval formula (2.3) applies to the functions $\chi_\lambda(t) \xi_{-\rho}(w^{-1}t\omega t^{-1})^{-1/2} e^{-(\lambda+\rho)H(t\omega t^{-1})}$ and $\chi_{\lambda'}(t) \xi_{-\rho}(w^{-1}t\omega t^{-1})^{-1/2} e^{-(\lambda'+\rho)H(t\omega t^{-1})}$.

First notice that these functions are locally bounded in t and so it suffices to consider their behavior for $t \to \infty$ in A_+ and $A_- = wA_+w^{-1}$. Next, from Lemma 3.1 we have

$$\begin{aligned} f_\lambda(t, \omega) &\sim e^{-\lambda \log t} \chi_{-\rho}(w^{-1}tw)^{-1/2} \chi_{-\rho}(t^{-1})^{-1/2} \xi_{-\rho}(w^{-1}\omega)^{-1/2} \\ &\quad \times e^{-(\lambda+\rho)H(t\omega t^{-1})} \end{aligned}$$

and

$$g'_\lambda(t,\omega) \sim e^{-\lambda' \log t} \chi_{-\rho}(w^{-1}tw)^{-1/2} \xi_{-\rho}(w^{-1}\omega)^{-1/2} e^{-(\lambda'+\rho)H(t\omega t^{-1})}$$

and using Lemma 3.2 and ignoring $\xi_{-\rho}(w^{-1}\omega)^{-1/2}$ we get

$$f_\lambda(t,\omega) \sim e^{-\lambda \log t} e^{-\rho \log t} e^{-(\lambda+\rho)H(t\omega t^{-1})}$$

and

$$g_{\lambda'}(t,\omega) \sim e^{-\lambda' \log t} e^{-\rho \log t} e^{-(\lambda'+\rho)H(t\omega t^{-1})}.$$

Now, if $t \in A_+$, since ω is in V', $\lim\limits_{t \underset{A_+}{\rightarrow} \infty} t\omega t^{-1} = 1$. Thus for $t \underset{A_+}{\rightarrow} \infty$, $e^{-(\lambda+\rho)H(t\omega t^{-1})}$ and $e^{-(\lambda'+\rho)H(t\omega t^{-1})}$ are bounded, while $e^{-(\lambda+\rho)\log t}$ and $e^{-(\lambda'+\rho)\log t}$ are integrable on A_+ because $\rho > \lambda$ and also $\rho > -\lambda'$.

Before we consider t in A_- we shall express f_λ and $g_{\lambda'}$ in another form. For x in V

$$H(x) = H(w^{-1}x) = H(v(w^{-1}x)) + \log a(w^{-1}x),$$

thus

$$H(t\omega t^{-1}) = H(v(w^{-1}t\omega t^{-1}) + \log a(w^{-1}t\omega t^{-1}).$$

Also, we have

$$\log a(w^{-1}t\omega t^{-1}) = \log a(w^{-1}\omega) - \log t + \log(w^{-1}tw)$$

and

$$v(w^{-1}t\omega t^{-1}) = w^{-1}tw\ v(w^{-1}\omega)w^{-1}t^{-1}w.$$

After discarding terms independent of t we have

$$f_\lambda(t,\omega) \sim e^{-(\lambda+\rho)\log t} e^{-(\lambda+\rho)\log(w^{-1}tw)} e^{(\lambda+\rho)\log t}$$

$$\times\, e^{-(\lambda+\rho)H(w^{-1}tw\ v(w^{-1}\omega)w^{-1}t^{-1}w)}$$

and

$$g_{\lambda'}(t,\omega) \text{ similarly with } \lambda \leftrightarrow \lambda'.$$

Simplifying gives

$$f_\lambda(t,\omega) \sim e^{-(\lambda+\rho)\log(w^{-1}tw)} e^{-(\lambda+\rho)H(w^{-1}tw\ v(w^{-1}\omega)w^{-1}t^{-1}w)}$$

and $g_{\lambda'}$ similarly with $\lambda \leftrightarrow \lambda'$.

Now let $t \in A_-$. Then $w^{-1}tw \in A_+$ and Harish-Chandra [H-C] has shown that if $h \in A_+, \mu > 0$ and $x \in V$ then

$$\mu(H(x) - H(hxh^{-1})) \leq 0.$$

So, with $\mu = \mathrm{Re}\ (\rho+\lambda)$ or $\mu = \mathrm{Re}\ (\rho+\lambda')$ we get

$$|e^{-(\lambda+\rho)H(w^{-1}tw\ v(w^{-1}\omega)w^{-1}t^{-1}w)}| \leq e^{-\mathrm{Re}(\lambda+\rho)H(w^{-1}\omega)}$$

and similarly for the λ' term.

Thus, ignoring terms bounded, for t in A_- we have

$$|f_\lambda(t,\omega)| \leq e^{-\mathrm{Re}(\lambda+\rho)\log(w^{-1}tw)}$$

and

$$|g_{\lambda'}(t,\omega)| \leq e^{-\mathrm{Re}(\lambda'+\rho)\log(w^{-1}tw)}.$$

Both of which are integrable on A_-. Hence, the Parseval relationship (1.3) can be applied to f, g.

Remark. We note that the only use of dim $\mathfrak{a} = 1$ was to write $A = \mathrm{cl}(A_+ \cup A_-)$. Otherwise, the estimates hold in general rank on $\mathfrak{a}_+$ and the opposite chamber.

§6 Computation of Bochner measure.

The function part of the Bochner measure, $m(\lambda, \upsilon)$, contains much information about the matrix coefficients of unitary induced representations. For complementary series the unitary structure on $\widetilde{\mathcal{H}}(1,\lambda)$ must be modified by the intertwining operator, however on the K-fixed vector the intertwining operator in the "compact picture" is a scalar. Thus we can use Theorem 5.1 for the spherical complementary series (σ trivial) as well as for the spherical principal series and the zonal spherical function. To get some understanding of $m(\lambda, \upsilon)$ we shall compute it, exactly in these cases.

We first recall some calculations from [H2]. The set Σ_+ consists of, at most, $\{\alpha, 2\alpha\}$. We let $p = \dim \mathfrak{g}_\alpha$ and $q = \dim \mathfrak{g}_{2\alpha}$ and choose the usual inner product on $\mathfrak{g}$. If X denotes elements of $\mathfrak{g}_{-\alpha}$ and Y those from $\mathfrak{g}_{-2\alpha}$, each x in V can be written uniquely $x = \exp X \exp Y$. The relevant formulae are ([H2] p.59)

$$\xi_{-\rho}(w^{-1}x) = [c^2|X|^4 + 4c|Y|^2]^{-1/4(p+2q)} \tag{6.1}$$

$$e^{\rho H(x)} = [(1+c|X|^2)^2 + 4c|Y|^2]^{1/4(p+2q)} \tag{6.2}$$

where $c^{-1} = 4(p+4q)$

As remarked in §3, $\mathfrak{v}' = \mathfrak{v}\backslash\{0\}$. It follows from Lemma 3.1 that the level set $\xi_{-\rho}(w^{-1}x) = 1$ provides a cross-section $A\backslash V' \to V'$. Also, being $\mathbb{R}$ rank 1 we express $\lambda = \lambda\rho$ with $|\mathrm{Re}\ \lambda| < 1$ in order to apply Theorem 5.1.

The K-fixed vector in $\widetilde{\mathcal{H}}(1,\lambda)$ is

$$f_\lambda(a,\omega) = e^{-\lambda\rho\log a} e^{-(\lambda\rho+\rho)H(a\omega a^{-1})}\xi_{-\rho}(w^{-1}a\omega a^{-1})^{-1/2}$$

$$= e^{-\lambda\rho\log a} e^{(\lambda\rho+\rho)H(a\omega a^{-1})} e^{-\rho\log a}\xi_{-\rho}(w^{-1}\omega)^{-1/2}.$$

Using $\xi_{-\rho}(w^{-1}\omega) = 1$ and Helgason's formula (6.2) gives

$$f_\lambda(a,\omega) = e^{-(\lambda\rho+\rho)\log a}[(1+c|e^{-\alpha\log a}X|^2)^2 + 4c|e^{-2\alpha\log a}Y||^2]^{-\frac{p+2q}{2}(1+\lambda)}$$

We set $s = e^{-\alpha\log a}$ and use $\rho = (p+2q)\alpha/2$ to get

$$f_\lambda(s,\omega) = \frac{s^{\frac{p+2q}{2}(1+\lambda)}}{[(1+cs^2|X|^2)^2 + 4cs^4|Y|^2]^{\frac{p+2q}{2}(1+\lambda)}}.$$

Next setting $t = s^2$ and using $1 = \xi_{-\rho}(w^{-1}x) = [c^2|X|^4 + 4c|Y|^2]$ gives

$$f_\lambda(t,\omega) = \frac{t^{\frac{p+2q}{4}(1+\lambda)}}{[1+2t\ c|X|^2+t^2]^{\frac{p+2q}{4}(1+\lambda)}}.$$

Finally, since $c|X|^2 \le 1$ we write it as $\cos\,\theta$. Thus, using (2.1), we have

$$\widetilde{f}(\upsilon\rho,\omega) = \int_A \chi_{i\upsilon\rho}(a)^{-1} f_\lambda(a,\omega)da$$

$$= \int_0^\infty \frac{t^{\frac{p+2q}{4}(1+\lambda-i\upsilon)}}{[1+2t\cos\theta+t^2]^{\frac{p+2q}{4}(1+\lambda)}}\frac{dt}{t}$$

as the integral to be evaluated for λ and similarly λ'. Let $a = \left(\frac{p+2q}{4}\right)(1+\lambda-i\upsilon) - 1$ and $b = \left(\frac{p+2q}{4}\right)(1+\lambda)$ and note that Re $a > -1$ and Re $b > 0$. We consider, under these circumstances, the integral

$$I = \int_0^\infty \frac{t^a}{[1+2t\cos\theta+t^2]^b}dt, \theta \in [0,\frac{\pi}{2}].$$

We factor the denominator and treat the integral as a contour integral. Thus,

$$\begin{aligned} I &= \int_0^\infty \frac{t^a}{(e^{i\theta}+t)^b(e^{-i\theta}+t)^b}dt \\ &= e^{-2i\ b\theta}e^{ia\theta}\int_0^\infty \frac{[e^{-i\theta}t]^a dt}{[1+e^{-i\theta}t]^b[e^{-2i\theta}+e^{-i\theta}t]^b}. \end{aligned}$$

After some elementary manipulations involving deformation of contours one arrives at

$$I = e^{i(a+1)\theta}(e^{2i\theta}-1)^{-b}\int_0^1 x^a(1-x)^{2b-a-2}(x-z)^{-b}dx.$$

With $v = z^{-1}$ this is found in [W-W], p. 293 to be

$$I = \frac{\Gamma(a+1)\Gamma(2b-a-1)}{\Gamma(2b)}e^{i(a+1)\theta}F(b, a+1; 2b; v),$$

using Gauss' hypergeometric function, provided $|v| < 1$. Now $|v| = |2\sin\ \theta| < 1$ means $0 \leq \theta < \frac{\pi}{6}$. To get the integral for $\theta \in [0, \frac{\pi}{2}]$ we analytically continue the hypergeometric function by the quadratic transformation ([BP])

$$F(\alpha, \beta; 2\beta; z) = (1-z)^{-\alpha/2}F(\frac{\alpha}{2}, \beta - \frac{\alpha}{2}; \beta + \frac{1}{2}; \frac{z^2}{4(z-1)})$$

or, finally,

$$I = \frac{\Gamma(a+1)\Gamma(2b-a-1)}{\Gamma(2b)}F(\frac{a+1}{2}, b - \frac{a+1}{2}; b + \frac{1}{2}; \sin^2\theta).$$

Recalling how a, b were defined we get

$$\widetilde{f}(v,\omega) = \frac{\Gamma(\frac{(p+2q)}{4}(1+\lambda-iv))\Gamma(\frac{(p+2q)}{4}(1+\lambda+iv))}{\Gamma(\frac{(p+2q)}{4}(2+2\lambda))} \tag{6.3}$$

$$\times F(\alpha, \beta; \gamma; \sin^2\theta) \text{ where}$$

$$\alpha = (\frac{p+2q}{4})(\frac{1}{2}+\frac{\lambda}{2}-\frac{iv}{2}), \beta = (\frac{p+2q}{4})(\frac{1}{2}+\frac{\lambda}{2}+\frac{iv}{2})$$

$$\gamma = (\frac{p+2q}{4})(1+\lambda)+\frac{1}{2}.$$

Simiarly, for λ'.

Some comments are in order. When there is no 2α root the $\cos\theta$ appears as 1 in the beginning and then the hypergeometric function is evaluated at zero, hence is 1 (a considerable simplification!). Next, after we had evaluated the integral, we happened across it in a table. But, as similar computations for prehomogeneous vector spaces of parabolic type are a possibility, we think it worth presenting. Finally, compressing the notation, the resultant integral formula for the spherical function φ_λ,

$$\varphi_\lambda(a) = \int_{\mathbb{R}} e^{-iv\log a} m(\lambda, v)\, dv \tag{6.4}$$

seems new to us even from the special function literature. The only analogous, though definitely different, integrals were obtained by Barnes [Ba].

§7. Applications.

In this section, we give four applications of the previous sections.

Positive Definite.

We have already remarked that $m(\lambda, \upsilon) \geq 0$ is necessary for unitarity of $\widetilde{U}$. Conversely, one might ask if it is sufficient, or, in other words, if the restriction to A are positive definite must $\widetilde{U}$ be unitary?

Proposition 7.1. *Let $\lambda = \lambda\rho$ and suppose $|Re\ \lambda| \leq 1$. If either (i) Re $\lambda = 0$ or (ii) Im $\lambda = 0$, then φ_λ is positive definite on A.*

Proof. First suppose $|\text{Re}\ \lambda| < 1$ for then we may use (6.3). Of course it suffices to show that $m(\lambda, \upsilon \geq 0)$. If Re $\lambda = 0$, then $\lambda' = \lambda$ and

$$m(\lambda, \upsilon) = \int\limits_{A \backslash V'} \widetilde{f}(\upsilon, \omega)\overline{\widetilde{f}(\upsilon, \omega)} dw \geq 0.$$

If Im $\lambda = 0$, then $\lambda' = -\lambda$. In (6.3) the parameters α, β in the hypergeometric function are complex conjugates, while γ is positive, so the function is positive. Also, the Γ-functions in the numerator are conjugates while the denominator is positive. Hence, $\widetilde{f}(\upsilon, \omega) > 0$ for λ and similarly for λ'. If $\lambda = \rho$, a careful look at the integral I shows the calculation is valid for λ and as just observed is a positive function. But for $\lambda' = -\rho$ we must compute the Fourier transform of 1, getting a Dirac measure. Thus, the Bochner measure for $\lambda = \rho$ is positive multiple of a Dirac measure at $\upsilon = 0$.

In [K] the unitarizable spherical principal series are determined and shown to be (i) Re $\lambda = 0$ or (ii) Im $\lambda = 0$ and $|\lambda| \leq \frac{p}{2}$ when $q = 0$ and when $q \neq 0$ one also has $\frac{p+2q}{2}$. Comparison with Proposition 7.1 suggests an extension problem.

Problem 1. *Let f be defined on G and suppose f is positive definite on A. Find sufficient conditions for f to be positive definite on G.*

If instead we restricted to a larger group there is an easy answer.

Lemma 7.2. *Let G be a locally compact group, K a closed subgroup $S \subseteq G$ with $G = KS$. Let f be defined on G and bi-K-invariant. Then, f is positive definite on G if and only if f is positive definite on S.*

Proof. It is enough to show f is p.d. on G. Let $x_1, \cdots, x_n$ be in G and say $x_i = k_i s_i$ Let $c_i, \cdots, c_n$ be complex numbers.

$$\begin{aligned} \sum_{i.j} f(x_i x_j^{-1}) \overline{c}_i c_j &= \sum_{i,j} f(k_i s_i s_j^{-1} k_j^{-1}) \overline{c_i} cj \\ &= \sum_{i,j} f(x_i s_j^{-1}) \overline{c_i} c_j \geq 0. \end{aligned}$$

In [F-K] the set $S = AK$ was used and the K-type expansion together with Bochner's theorem on the compact group K gave another proof of Kostant's theorem. Alternatively $S = MAN$ leads to the use of Kirillov theory on N together with a Bochner theorem for nilpotent groups.

Harish-Chandra expansion.

We shall obtain from the integral formula (6.4) another derivation of Harish-Chandra's expansion for the elementary spherical function $\varphi_\lambda(a)$ but without the use of any differential equations. Fix H $\in \mathfrak{a}$ of norm one, and set $a_t = exptH$. Recall that the level set $\Omega = \{\xi_{-\rho}(w^{-1}x = 1\}$ is a cross-section for $A\backslash V'$. From (5.1)

$$\varphi_\lambda(a_t) = \int\limits_{\mathbb{R}} e^{-i\upsilon t} \int\limits_{\Omega} < \widetilde{f}(\upsilon,\omega), \widetilde{g}(\upsilon,\omega) >_\sigma d\omega d\upsilon$$

here $\widetilde{f}(\upsilon,\omega)$ is given by (6.3). But first, we need

Lemma 7.3. *If $\lambda = \lambda\rho$ with $|Re\ \lambda| < 1$ then the Bochner measure is of the form*

$$m(\lambda,\upsilon) = \Upsilon(\lambda,\upsilon)\Upsilon(-\lambda,\upsilon)h(\lambda,\upsilon)$$

where h is entire in $\upsilon \in \mathfrak{a}_c$ and

$$\Upsilon(\lambda,\upsilon) = \frac{\Gamma(r(1+\lambda-i\upsilon))\Gamma(r(1+\lambda+i\upsilon))}{\Gamma(r(2+2\lambda))}$$

here $r = (p+2q)/4$.

Proof. From (6.3) we define $\Upsilon(\lambda,\upsilon)$ to be the gamma functions appearing there. Next, simply observe that the complex conjugate of (4.5) for λ' simply takes $\lambda \to -\lambda$. We must consider the integral over $A\backslash V'$ of the product of two hypergeometric functions. But, the υ-dependence is only in the α and β parameters in which the hypergeometric function is entire (note that $\mathrm{Re}(\Upsilon - \alpha - \beta) > 0$) and as above, the complex conjugate can be absorbed as a change of sign since $A\backslash V'$ may be viewed as the compact set $\xi_{-\rho}(w^{-1}x) = 1$ we have the resulting integral, $h(\lambda,\upsilon)$, entire in $\upsilon \in \mathfrak{a}_c$.

We illustrate the computation by giving the details for $\mathrm{Sl}(2,\mathbb{R})$ and λ imaginary. In order to compare the result to ones in the literature we change notation slightly. Henceforth λ imaginary will be written as $i\lambda := i\lambda\alpha$, and similarly in the integral we replace υ with $\upsilon\alpha$ and tH with the co-root $t\check{H}_\alpha$. In this case the integral becomes with $r = 1/4$.

(7.1)

$$\varphi_\lambda(a_t) = c\int\limits_{\mathbb{R}} e^{-i\upsilon t}\frac{\Gamma(r(1+i(\frac{\lambda}{2}-\frac{\upsilon}{2})))\Gamma(r(1+i(\frac{\lambda}{2}+\frac{\upsilon}{2})))}{\Gamma(r(2+4\lambda))}\frac{\Gamma(r(1-i(\frac{\lambda}{2}+\frac{\upsilon}{2})))\Gamma(r(1-i(\frac{\lambda}{2}-\frac{\upsilon}{2})))}{\Gamma(r(2-4\lambda))}d\upsilon$$

The integral consists of terms of the form $\Gamma(r \pm i(r/2)(\lambda \pm \upsilon))$ and these functions have poles at $\pm\lambda \pm i(2k+1/2)$. For $\lambda \neq 0$ all poles of the integrand are simple and one easily sees that the residues are of $\Gamma(z)$ when $z = -k$. To compute $\mathrm{Res}_{-k}\Gamma(z)$ use the duplication formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ and one obtains

$$\mathrm{Res}_{-k}\Gamma(z) = \frac{(-1)^k \pi}{\Gamma(k+1)}.$$

To evaluate the other terms one needs to simplify $\Gamma(-k+i\lambda)$. Again, a use of the duplication formula and the fact that $\sin \pi i\lambda = \frac{(-1)\pi}{i\lambda\Gamma(i\lambda)\Gamma(-i\lambda)}$ gives

$$\Gamma(-k+i\lambda) = \frac{(-1)^{k+1} i\lambda\Gamma(i\lambda)\Gamma(-i\lambda)}{\Gamma(1+k-i\lambda)}.$$

We evaluate (7.1) by an application of Cauchy's theorem and a rectilinear contour from -R to R to R+iR to -R+iR to -R. It is routine to show the contributions other than along -R to R tend to zero as $R \to \infty$. The result is

$$\begin{aligned} \varphi_\lambda(a_t) = {} & \frac{\Gamma(i\lambda)}{\Gamma(1/2+i\lambda)} \sum_{k=0}^{\infty} e^{-i\lambda t} e^{-(2k+1/2)t} \frac{\Gamma(k+1/2)}{\Gamma(k+1)} \frac{\Gamma(1/2+k-i\lambda)}{\Gamma(1/2-i\lambda)} \frac{\Gamma(1-i\lambda)}{\Gamma(1+k+i\lambda)} \\ & + \frac{\Gamma(-i\lambda)}{\Gamma(1/2-i\lambda)} \sum_{k=0}^{\infty} e^{i\lambda t} e^{-(2k+1/2)t} \frac{\Gamma(k+1/2)}{\Gamma(k+1)} \frac{\Gamma(1/2+k+i\lambda)}{\Gamma(1/2+i\lambda)} \frac{\Gamma(1+i\lambda)}{\Gamma(1+k-i\lambda)}. \end{aligned}$$

Remarks. (1) It is curious that this derivation uses in no explicit way the differential equation for the spherical function, in contrast to all other derivations of it.

(2) Obviously the same computation works for SO $(n,1)$ using a different value of "r". The computation for the other rank one groups is complicated by having to evaluate

$$\int_0^{\pi/2} |F(\alpha, \beta; \gamma; sin^2(\theta))|^2 d\theta.$$

Our only progress to obtain a clean formula here was to expand one of the hypergeometric functions and use [B] p.399 to evaluate each of the resulting integrals; thereby obtaining an infinite series in quotients of Γ functions at various arguments. This was sufficient to obtain the "c" functions and the leading term.

Holomorphic extension.

Again, we illustrate the method by presenting the details solely for $\mathrm{Sl}(2,\mathbb{R})$. We begin with (7.1).

$$\varphi_\lambda(a_t) = c \int_{\mathbb{R}} e^{-i\upsilon t} \frac{\Gamma(r(1+i(\frac{\lambda}{2}-\frac{\upsilon}{2})))\Gamma(r(1+i(\frac{\lambda}{2}+\frac{\upsilon}{2})))}{\Gamma(r(2+4\lambda))} \frac{\Gamma(r(1-i(\frac{\lambda}{2}+\frac{\upsilon}{2})))\Gamma(r(1-i(\frac{\lambda}{2}-\frac{\upsilon}{2})))}{\Gamma(r(2-4\lambda))} d\upsilon.$$

We are interested in the Paley-Wiener phenomenon, i.e. exponential decay of the Fourier transform is equivalent to a holomorphic extension of the function to a tube. The case $\lambda = 0$ contains all the essential technicalities; so we present that. Hence consider

$$I = \int_{\mathbb{R}} e^{i\upsilon x} |\Gamma(\frac{1}{4} - \frac{i}{2}\upsilon)|^2 |\Gamma(\frac{1}{4} + \frac{i}{2}\upsilon)|^2 d\upsilon.$$

Since $\overline{\Gamma(u+iv)} = \Gamma(u - iv)$ the Γ functions in the integral can be rewritten as

$$[\Gamma(\frac{1}{4} + \frac{i}{2}\upsilon)\Gamma(\frac{1}{4} - \frac{i}{2}\upsilon)]^2.$$

Take the logarithm and use Binet-Stirling to get

$$\begin{aligned}\log\Gamma(\frac{1}{4} + \frac{i}{2}\upsilon) + \log\Gamma(\frac{1}{4} - \frac{i}{2}\upsilon) = &-\frac{1}{2}\log|\frac{1}{4} + \frac{i}{2}\upsilon| - \frac{\upsilon}{2}\arg(\frac{1}{4} + \frac{i}{2}\upsilon) \\ &+ \frac{\upsilon}{2}\arg(\frac{1}{4} - \frac{i}{2}\upsilon) - \frac{i}{4}\arg(\frac{1}{4} + \frac{i}{2}\upsilon) \\ &- \frac{i}{4}\arg(\frac{1}{4} - \frac{i}{2}\upsilon) + O(1).\end{aligned}$$

Now for $\upsilon > 0$ we have $\arg(\frac{1}{4} \pm \frac{i}{2}\upsilon) \sim \pm\pi/2 \mp \frac{1}{2\upsilon}, \upsilon \to \infty$, while for $\upsilon < 0$ we have $\arg(\frac{1}{4} \pm \frac{i}{2}\upsilon) \sim \mp\pi/2 \pm \frac{1}{2|\upsilon|}, \upsilon \to \infty$. Then

$$\arg(\frac{1}{4} + \frac{i}{2}\upsilon) - \arg(\frac{1}{4} - \frac{i}{2}\upsilon) \sim \pm\pi \mp \frac{1}{|\upsilon|},\ \upsilon < 0(\,resp.\,\upsilon > 0),$$

while

$$\arg(\frac{1}{4} + \frac{i}{2}\upsilon) + \arg(\frac{1}{4} - \frac{i}{2}\upsilon) = o(1), \upsilon \to \infty.$$

Then

$$\begin{aligned}\log\Gamma(\frac{1}{4} + \frac{i}{2}\upsilon) + \log\Gamma(\frac{1}{4} - \frac{i}{2}\upsilon) &\sim \log|\frac{1}{4} + \frac{i}{2}\upsilon|^{-1/2} - \frac{\upsilon}{2}[\pm\pi \mp \frac{1}{|\upsilon|}] + O(1) \\ &\sim \log|\frac{1}{4} + \frac{i}{2}\upsilon|^{-1/2} - |\upsilon|\pi/2 + O(1).\end{aligned}$$

Consequently

$$[\Gamma(\frac{1}{4} + \frac{i}{2}\upsilon)\Gamma(\frac{1}{4} - \frac{i}{2}\upsilon)]^2 \sim c\frac{e^{-\pi|\upsilon|}}{\sqrt{\frac{1}{16} + \frac{\upsilon^2}{4}}}.$$

So the spherical function holomorphically extends in "x" to "$x + iy$". Moreover, one can detect the nature of the singularity. Indeed, taking $x = 0$ and writing $y = \pi - \varepsilon$, we obtain the Laplace transform

$$\int_{|\upsilon|>R} e^{-\varepsilon\upsilon}\frac{d\upsilon}{\sqrt{\frac{1}{16} + \frac{\upsilon^2}{4}}}$$

which is easily seen to be $\sim \ln \varepsilon$. Both these results can be found in [KS-I] with a different proof. In fact, the preceeding computation was used as independent verification of the computations therein.

Remarks. The extension to $\lambda \neq 0$ presents no difficulties, nor does the extension to $SO(n,1)$. The remaining rank 1 groups can be done this way; albeit with additional work. As usual, the sticky point is (5.2) which here is the integral over Ω of $F(2a, 2b; a+b+\frac{1}{2}; \sin^2\frac{\theta}{2})$. However we need only estimate - not evaluate - it. After some transformations on the parameters of the hypergeometric function one obtains (up to easily estimated factors) the Legendre function $P_n^m(cos\varphi)$ where $n \in \mathbb{C}$, $|n| \to \infty$. For this the asymptotics in [Wa] p.291 are adequate to obtain estimates which allow one to detect the size of the holomorphic extension, and the nature of the singularity. We omit the details because there is a detailed proof of the result in [KS-I].

Spectral F_f.

We begin by recalling properties of the Abel transform as defined by Harish-Chandra. For $f \in C_c(G//K)$ define F_f by

$$F_f(a) = e^{\rho loga} \int_N f(an)dn.$$

The Abel transform $\mathcal{A} : f \to F_f$ has several interesting properties of which we highlight only those immediately relevant.

$$\mathcal{A} : C_c(G//K) \longrightarrow C_c^W(A) \tag{7.2}$$

where $C_c^W(A)$ denotes the compactly supported Weyl group invariant functions on A.

(7.3) $\mathcal{A}$ is continuous in the usual topologies.

(If one uses the Harish-Chandra-Schwarz space, then an analogous version of both (7.2) and (7.3) are valid.) Hence there exists $\mathcal{A}' : C_c^W(A)' \longrightarrow C_c(G//K)'$ satisfying for $f \in C_c(G//K)$ and $T \in C_c^W(A)'$

$$< \mathcal{A}f, T > = < f, \mathcal{A}'T > .$$

Let φ_λ, λ pure imaginary, be the elementary spherical function. Then we have that $\varphi_\lambda \in C_c(G//K)'$; similarly we have the averaged character $\frac{1}{|W|}\sum_W e^{w\lambda(\cdot)} \in C_c^W(A)'$. Using $\widetilde{f}$ to denote the spherical transform of f we recall the familiar identity

$$\widetilde{f}(\lambda) = \widehat{F_f}(\lambda) \tag{7.4}$$

The duality relationship

$$< \mathcal{A}f, \frac{1}{|W|}\sum_W e^{w\lambda(\cdot)} > = < f, \mathcal{A}'(\frac{1}{|W|}\sum_W e^{w\lambda(\cdot)}) >,$$

combined with the fact that $\mathcal{A}'$ is 1-1 give the identity

$$\varphi_\lambda(\cdot) = \mathcal{A}' \frac{1}{|W|} \sum_W e^{w\lambda(\cdot)}. \tag{7.5}$$

Indeed this relationship can be read in reverse, so that (7.4) and (7.5) are equivalent. As an aside, in [S-T] the specific form, in the case of real rank 1, of $\mathcal{A}'$ as an integral operator in (7.5) was used to derive an expansion for the elementary spherical functions in terms of spherical functions for the associated Cartan motion groups.

Now take $f \in \mathcal{S}(G//K)$ or just in $C_c(G//K)$ but with sufficient decay of its spherical Fourier transform, $\widetilde{f}$. Recall the Bochner measure $m(\lambda, v)$. Define $F^s_{\widetilde{f}}(v)$ by

$$F^s_{\widetilde{f}}(v) = \int_{i\mathfrak{a}^*} \widetilde{f}(\lambda) m(\lambda, v) \frac{d\lambda}{|c(\lambda)|^2}. \tag{7.6}$$

Recall the inversion formula for the spherical transform

$$f(a) = \int_{i\mathfrak{a}^*} \widetilde{f}(\lambda) \varphi_\lambda(a) \frac{d\lambda}{|c(\lambda)|^2}.$$

Then an elementary application of the Fubini theorem and the Euclidean Fourier inversion theorem gives

$$f(a) = F^{s\vee}_{\widetilde{f}}(a) \tag{7.7}$$

which is to be contrasted with (7.4).

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